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EQUIVARIANT BORDISM AND SMITH THEORY

BY
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Abstract. The relationship between equivariant bordism and Smith homology theory on the category of pairs with involution is studied.

1. **Introduction.** The object of this paper is to analyze the relationship between the homology theories given by equivariant bordism $\mathfrak{N}_*^{Z_2}(X, A, T)$ and Smith homology theory $H_*^{Z_2}(X, A, T; Z_2)$ on the category of pairs with involution. These theories are related in much the same way as ordinary bordism and homology are related. (See for example [1, §8 and §17].)

In §2, basic definitions will be given, and then it will be shown that assigning to a bordism element $f: (M, \partial M, S) \to (X, A, T)$ the image of the fundamental Smith theory class of $(M, \partial M, S)$ defines a natural isomorphism

$$\tilde{\mu}: \mathfrak{N}_*^{Z_2}(X, A, T) \otimes_{\mathfrak{N}_*^{Z_2}} Z_2 \xrightarrow{\cong} H_*^{Z_2}(X, A, T; Z_2).$$

It should be noted, however, that one cannot have the other isomorphism relation (valid for ordinary bordism)

$$\mathfrak{N}_*^{Z_2}(X, A, T) \cong H_*^{Z_2}(X, A, T; Z_2) \otimes_{Z_2} \mathfrak{N}_*^{Z_2}$$

for all (X, A, T) since Z_2 bordism is not in general a free $\mathfrak{N}_*^{Z_2}$ module. (If $X=2$ points interchanged by T and $A=\varnothing$, $\mathfrak{N}_*^{Z_2}(X, A, T) \cong \mathfrak{N}_*$, while $H_*^{Z_2}(X, A, T) \otimes_{Z_2} \mathfrak{N}_*^{Z_2} \cong Z_2 \otimes_{Z_2} \mathfrak{N}_*^{Z_2} \cong \mathfrak{N}_*^{Z_2}$, and these are not isomorphic as graded groups.)

In §3, it will be shown that the composite

$$\mu \circ \tau_*: \mathfrak{N}_*^{Z_2}(X, A, T) \to H_*^{Z_2}(X \times BO, A \times BO, T \times S; Z_2)$$

is monic, where $\tau_*: \mathfrak{N}_*^{Z_2}(X, A, T) \to \mathfrak{N}_*^{Z_2}(X \times BO, A \times BO, T \times S)$ sends $f: (M, \partial M, Q) \to (X, A, T)$ to $f \times \tau: (M, \partial M, Q) \to (X \times BO, A \times BO, T \times S)$ with τ equivariantly classifying the stable tangent bundle of M. Thus, the dual Smith cohomology groups

$$H_{Z_2}^*(X \times BO, A \times BO, T \times S; Z_2)$$

provide characteristic numbers which determine the bordism class. There has been much recent interest in equivariant characteristic numbers (see [5], for example). Much of the effort has been devoted to more difficult cases using modern methods

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with equivariant Thom classes and such (in for example the work of T. tom Dieck or G. Hamrick). The approach taken here is much more naive.

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2. **The representation theorem.** Let X be a simplicial complex $T\colon X \to X$ a simplicial involution ($T^2 = \text{identity}$) and $A \subset X$ an invariant subcomplex ($TA \subset A$). It will be assumed that X is "finely" triangulated so that the fixed set F of T on X is a subcomplex and the orbit map $\pi\colon X \to X/Z_2$ is simplicial (E. E. Floyd [3] shows that this may be accomplished by taking the second barycentric subdivision).

Letting $C(X) \otimes Z_2$ denote the chains of X with mod 2 coefficients, one has an induced chain map $T_\#\colon C(X) \otimes Z_2 \to C(X) \otimes Z_2$ with $T_\# \circ T_\# = 1$, and one lets $C^0(X)$ denote the subgroup of chains invariant under $T_\#$ ($T_\#\sigma = \sigma$). Since the boundary operator commutes with $T_\#$, one has induced a boundary homomorphism $\partial\colon C^0(X) \to C^0(X)$, making $(C^0(X), \partial)$ a complex. One lets $C^0(X, A) = C^0(X)/C^0(A)$, and ∂ makes this into a chain complex. The *Smith homology groups* of (X, A, T), $H_*^{Z_2}(X, A, T; Z_2)$ are then defined to be the homology groups of the complex $(C^0(X, A), \partial)$.

Now consider a chain $\sigma = \sum a_i \Delta^i$, where Δ^i are n-simplices and $a_i \in Z_2$. The chain σ is invariant under $T_\#$ if and only if, for each n-simplex Δ, the coefficients of Δ and $T_\#\Delta$ in σ are the same. Thus σ is a sum of terms $\Delta + T_\#\Delta$ (if $T_\#\Delta \neq \Delta$) and terms Δ (with $T_\#\Delta = \Delta$), so that $C^0(X)$ decomposes into a direct sum $U \oplus V$, where U is spanned by the $\Delta + T_\#\Delta$ ($T_\#\Delta \neq \Delta$) and V is spanned by the Δ with $T_\#\Delta = \Delta$. Clearly $\partial(\Delta + T_\#\Delta)$ is a sum of terms of the same form for if a face Δ' of Δ is fixed by $T_\#$, Δ' occurs with coefficient $2 \equiv 0$ in $\partial(\Delta + T_\#\Delta)$, while if $T_\#\Delta = \Delta$, then by the "fineness" of the triangulation, each vertex of Δ must be fixed so each face of Δ is fixed by $T_\#$. Thus $C^0(X)$ is the direct sum of the subcomplexes U and V, inducing a decomposition of $C^0(X, A)$ and $H_*^{Z_2}(X, A, T; Z_2)$.

Now let $\rho\colon C(X/Z_2) \otimes Z_2 \to C(X) \otimes Z_2$ be defined as follows. If Δ' is a simplex of X/Z_2, let Δ be a simplex of X with $\pi\Delta = \Delta'$, and let $\rho(\Delta') = \Delta + T_\#\Delta$, which is well defined since Δ and $T_\#\Delta$ are the two possible lifts of Δ'. Clearly ρ is a chain map with image $U \subset C^0(X)$ and kernel $C(F) \subset C(X/Z_2)$. Thus ρ induces an isomorphism of $C(X/Z_2)/C(F) + C(A/Z_2) = C(X/Z_2, A/Z_2 \cup F)$ with the U summand of $C^0(X, A)$, and the U summand of $H_*^{Z_2}(X, A, T; Z_2)$ is identified with $H_*(X/Z_2, A/Z_2 \cup F; Z_2)$. Clearly the inclusion $i\colon F \to X$ identifies $C(F)$ with V, and so the V summand of $H_*^{Z_2}(X, A, T; Z_2)$ is identified with $H_*(F, F \cap A; Z_2)$. This gives the standard result

THEOREM 2.1. $H_*^{Z_2}(X, A, T; Z_2) \cong H_*(X/Z_2, A/Z_2 \cup F; Z_2) \oplus H_*(F, F \cap A; Z_2)$.

By using Čech [6], [4] or singular [2] methods to obtain a complex, this may be extended to topological pairs with involution (X, A, T). Similarly using

Hom $(C(X); Z_2)$ one may form Smith cohomology groups dual to the homology groups.

Being given a compact differentiable manifold with boundary M^n with differentiable involution S, one may triangulate M "finely" so that S is simplicial (by triangulating M/Z_2 and lifting the triangulation). Clearly the fundamental cycle $\mu = \sum \Delta^i$, the sum of all n-simplices, is then an invariant chain, defining a fundamental class $[M, \partial M, S] \in H_n^{Z_2}(M, \partial M, S; Z_2)$. This lifts $[M, \partial M] \in H_n(M, \partial M; Z_2)$ back along the forgetful homomorphism from Smith theory to ordinary homology.

One then has a natural homorphism

$$\mu: \mathfrak{N}_*^{Z_2}(X, A, T) \to H_*^{Z_2}(X, A, T; Z_2)$$

assigning to the equivariant bordism element $f: (M, \partial M, S) \to (X, A, T)$ the class $f_*[M, \partial M, S]$.

If $b = (B, U) \in \mathfrak{N}_*^{Z_2}$ and $\alpha = (M, \partial M, S, f) \in \mathfrak{N}_*^{Z_2}(X, A, T)$, their product $b\alpha = (B \times M, B \times \partial M, U \times S, f \circ \pi_M) \in \mathfrak{N}_*^{Z_2}(X, A, T)$ is sent by μ to

$$f_* \circ \pi_{M_*}[B \times M, B \times \partial M, U \times S].$$

Since M/Z_2 and its fixed set are complexes of dimension at most that of M, Theorem 1 shows that $\pi_{M_*}[B \times M, B \times \partial M, U \times S] = 0$ if the dimension of B is positive. If the dimension of B is zero, with B consisting of k points, then $\pi_M: B \times M \to M$ is a k-fold cover and π_{M_*} sends the fundamental cycle of $B \times M$ to k times that of M. Thus $\mu(b\alpha) = k\mu(\alpha) = \varepsilon(b) \cdot \mu(\alpha)$, where $\varepsilon: \mathfrak{N}_*^{Z_2} \to Z_2$ is the augmentation to $\mathfrak{N}_*^{Z_2} \simeq \mathfrak{N}_0 \simeq Z_2$.

Thus μ induces a natural homomorphism

$$\tilde{\mu}: \mathfrak{N}_*^{Z_2}(X, A, T) \otimes_{\mathfrak{N}_*^{Z_2}} Z_2 \to H_*^{Z_2}(X, A, T; Z_2).$$

PROPOSITION 2.1. *$\tilde{\mu}$ is epic.*

Proof. It suffices to show that $\tilde{\mu}$ maps onto each of the summands of $H_*^{Z_2}(X, A, T; Z_2)$.

If $\alpha \in H_*(F, F \cap A; Z_2)$, there is an ordinary bordism class $f: (N, \partial N) \to (F, F \cap A)$ with $f_*[N, \partial N] = \alpha$. Letting 1 denote the trivial involution on N, $f: (N, \partial N, 1) \to (X, A, T)$ is an equivariant bordism element with $f_*[N, \partial N, 1] = \alpha$.

If $\alpha \in H_*(X/Z_2, A/Z_2 \cup F; Z_2)$, there is an ordinary bordism element $f: (N, \partial N) \to (X/Z_2, A/Z_2 \cup F)$ with $f_*[N, \partial N] = \alpha$. Using excision and a neighborhood of F which retracts by a deformation to F, one may find a double cover $\tilde{N} \xrightarrow{\pi'} N$, with covering involution S and an equivariant map $\tilde{f}: \tilde{N} \to X$ with the diagram

$$\begin{array}{ccc} \tilde{N} & \xrightarrow{\tilde{f}} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{f} & X/Z_2 \end{array}$$

commutative. Then $\tilde{f}(\partial N) \subset A \cup F$ and by applying excision methods to a neighborhood of A, one may after a small equivariant deformation if needed find a decomposition of ∂N into two invariant submanifolds $\partial N = N_0 \cup N_1$ with $N_0 \cap N_1 = \partial N_0 = \partial N_1$ with $\tilde{f}(N_0) \subset A, \tilde{f}(N_1) \subset F$. Let $\hat{N}$ be the quotient of $\tilde{N}$ obtained by identifying x with Sx for $x \in N_1$, with $\hat{f}: \hat{N} \to X$ induced by $\tilde{f}$, $\hat{\pi}: \hat{N} \to N$ induced by π' and with involution $\hat{S}$ induced by S. Then $\hat{f}: (\hat{N}, \partial\hat{N}, S) \to (X, A, T)$ is an equivariant bordism element and $\hat{f}_{\#}$ sends the fundamental cycle of $\hat{N}$ to the lift by ρ of $f_{\#}\mu$ where μ is the fundamental cycle of N. Then $\hat{f}_*[\hat{N}, \partial\hat{N}, S]$ is α in $H_*^{Z_2}(X, A, T)$.

Applying an excision to identify $(A \cup F, A)$ with $(F, F \cap A)$, the bordism exact sequence of the triple $(X, A \cup F, A)$ gives an exact sequence

$$\mathfrak{N}_*^{Z_2}(F, F \cap A, 1) \xrightarrow{i} \mathfrak{N}_*^{Z_2}(X, A, T) \xrightarrow{j} \mathfrak{N}_*^{Z_2}(X, A \cup F, T)$$

$$\partial: \mathfrak{N}_*^{Z_2}(X, A \cup F, T) \to \mathfrak{N}_*^{Z_2}(F, F \cap A, 1)$$

and one has

LEMMA 2.1. *The homomorphism ∂ is zero.*

Proof. (1) Applying the fixed point homomorphism

$$F: \mathfrak{N}_*^{Z_2}(X, A, T) \to \textstyle\bigoplus_{k=0}^{*} \mathfrak{N}_{*-k}(F \times BO_k, (F \cap A) \times BO_k)$$

of [7, §3], $F \circ i$ is a monomorphism onto a direct summand complementary to the $k=1$ term, so i is monic, or ∂ is zero.

(2) Any class $\alpha \in \mathfrak{N}_*^{Z_2}(X, A \cup F, T)$ may be represented by a map $f: (N, \partial N, S) \to (X, A \cup F, T)$, and by excising a neighborhood of the fixed set of N, one may suppose S is free. By excision and a small deformation of f one may suppose $\partial N = N_0 \cup N_1$, $\partial N_0 = \partial N_1 = N_0 \cap N_1$ with $f(N_0) \subset A$, $f(N_1) \subset F$. Let $\bar{N}$ be the quotient of N by identifying x with Sx for $x \in N_1$, with $\bar{f}: \bar{N} \to X$ and $\bar{S}: \bar{N} \to \bar{N}$ induced by f and S. Then $\bar{f}: (\bar{N}, \partial\bar{N}, \bar{S}) \to (X, A, T)$ has image α under j.

Note. This defines a splitting, identifying $\mathfrak{N}_*^{Z_2}(X, A \cup F, T)$ with the bordism classes for which the fixed point set has codimension 1.

Special Note. The splitting $\varphi: \mathfrak{N}_*^{Z_2}(X, A \cup F, T) \to \mathfrak{N}_*^{Z_2}(X, A, T)$ just constructed is only a splitting as $\mathfrak{N}_*$ modules and not as $\mathfrak{N}_*^{Z_2}$ modules (see [7, p. 57, note (4)]). However, being given $f: (N, \partial N, S) \to (X, A \cup F, T)$ representing α with S free and $\partial N = N_0 \cup N_1, f(N_0) \subset A, f(N_1) \subset F$, as above, and $m = [M, U] \in \mathfrak{N}_*^{Z_2}$, $\varphi(m\alpha) - m\varphi(\alpha)$ is the class of

$$g: M \times I \times N_1/\{(m, 0, n) \sim (Um, 0, Sn), (m, 1, n) \sim (m, 1, Sn)\} \to F$$

by $g(m, t, n) = f(n)$ and with involution induced by $U \times 1 \times S$. In particular, $f: (N_1, \partial N_1, S) \to (F, F \cap A, 1)$ is a free Z_2 bordism element and is a sum of products $u \otimes v$ with $u \in \mathfrak{N}_*^{Z_2}(\text{Free})(\text{point})$ and $v \in \mathfrak{N}_*(F, F \cap A)$. Crossing with $M \times I$ and acting as above sends this to a term $u' \otimes v$ with $u' \in \mathfrak{N}_*^{Z_2}$, and since $\dim I > 0$, $\dim u' > 0$. Thus $\varphi(m\alpha) - m\varphi(\alpha)$ is decomposable in the $\mathfrak{N}_*^{Z_2}$ module structure, so that φ induces a splitting

$$\varphi': \mathfrak{N}_*^{Z_2}(X, A \cup F, T) \otimes_{\mathfrak{N}_*^{Z_2}} Z_2 \to \mathfrak{N}_*^{Z_2}(X, A, T) \otimes_{\mathfrak{N}_*^{Z_2}} Z_2$$

for

$$j': \mathfrak{N}_*^{Z_2}(X, A, T) \otimes_{\mathfrak{N}_*^{Z_2}} Z_2 \to \mathfrak{N}_*^{Z_2}(X, A \cup F, T) \otimes_{\mathfrak{N}_*^{Z_2}} Z_2.$$

PROPOSITION 2.2. *$\bar{\mu}$ is an isomorphism.*

Proof. Tensoring the short exact sequence of the lemma with Z_2 and applying $\tilde{\mu}$ gives a commutative diagram

$$\begin{array}{ccccccccc}
Q \longrightarrow & \mathfrak{N}_*^{Z_2}(F, F \cap A) \otimes Z_2 & \xrightarrow{i'} & \mathfrak{N}_*^{Z_2}(X, A) \otimes Z_2 & \xrightarrow{j'} & \mathfrak{N}_*^{Z_2}(X, A \cup F) \otimes Z_2 & \longrightarrow 0 \\
& a \downarrow \tilde{\mu} & & b \downarrow \tilde{\mu} & & c \downarrow \tilde{\mu} & \\
0 \longrightarrow & H_*^{Z_2}(F, F \cap A) & \xrightarrow{i_*} & H_*^{Z_2}(X, A) & \xrightarrow{j_*} & H_*^{Z_2}(X, A \cup F) & \longrightarrow 0 \\
& \wr\| & & & & \wr\| & \\
& H_*(F, F \cap A) & & & & H_*(X/Z_2, A/Z_2 \cup F) &
\end{array}$$

in which the bottom exact sequence is split and gives the direct sum decomposition of Theorem 2.1, with extraneous data being dropped from the notation, and with Q being a Tor-term.

First, a is an isomorphism. Since $\mathfrak{N}_*^{Z_2}$ is a free $\mathfrak{N}_*$ module, the multiplication $\mathfrak{N}_*^{Z_2} \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(F, F \cap A) \to \mathfrak{N}_*^{Z_2}(F, F \cap A)$ is an isomorphism, being an isomorphism of homology theories on the category of pairs (with trivial action). Thus $\mathfrak{N}_*^{Z_2}(F, F \cap A) \otimes Z_2$ is isomorphic to

$$Z_2 \otimes_{\mathfrak{N}_*^{Z_2}} \mathfrak{N}_*^{Z_2} \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(F, F \cap A) \cong Z_2 \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(F, F \cap A) \cong H_*(F, F \cap A),$$

and these isomorphisms are just $\tilde{\mu}=a$. In particular, $i_*a=bi'$ is monic, so i' is monic, and the top sequence is short exact.

Next, c is an isomorphism. First, the pair $(X, A \cup F)$ is relatively free as a Z_2 pair, so $\mathfrak{N}_*^{Z_2}(X, A \cup F)$ is isomorphic to the free bordism group $\mathfrak{N}_*^{Z_2}(\text{Free})(X, A \cup F)$. Assigning to a free bordism element $f: (N, \partial N, S) \to (X, A \cup F, T)$ the map $\bar{f}: (N/Z_2, \partial N/Z_2) \to (X/Z_2, A/Z_2 \cup F)$ gives an $\mathfrak{N}_*$ module isomorphism

$$\rho: \mathfrak{N}_*^{Z_2}(X, A \cup F) \to \mathfrak{N}_*(X/Z_2, A/Z_2 \cup F).$$

Clearly $\bar{f}_*[N/Z_2, \partial N/Z_2]$ in $H_*(X/Z_2, A/Z_2 \cup F)$ represents $\mu(N, \partial N, S, f)$ under the identification with the Smith group, and hence

$$\hat{\mu}: \mathfrak{N}_*^{Z_2}(X, A \cup F) \otimes_{\mathfrak{N}_*} Z_2 \to H_*^{Z_2}(X, A \cup F)$$

induced by μ is an isomorphism. Letting

$$\theta: \mathfrak{N}_*^{Z_2}(X, A \cup F) \otimes_{\mathfrak{N}_*} Z_2 \to \mathfrak{N}_*^{Z_2}(X, A \cup F) \otimes_{\mathfrak{N}_*^{Z_2}} Z_2$$

be the quotient, θ and $\tilde{\mu}$ are epic while $\hat{\mu}=\tilde{\mu} \circ \theta$ is an isomorphism. Thus θ and $c=\tilde{\mu}$ are isomorphisms.

Note. This indicates that free bordism as $\mathfrak{N}_*^{Z_2}$ module essentially is given by the $\mathfrak{N}_*$ module structure: This is not precise, however.

Finally, since a and c are isomorphisms, so is b by the five lemma, which gives the proposition.

3. **The characteristic number theorem.** Let $\boldsymbol{R}^\infty \oplus \boldsymbol{R}^\infty$ denote the countable direct sum of copies of $\boldsymbol{R}$, with involution $s: \boldsymbol{R}^\infty \oplus \boldsymbol{R}^\infty \to \boldsymbol{R}^\infty \oplus \boldsymbol{R}^\infty : (x, y) \to (x, -y)$, and let BO_n be the space of n-dimensional subspaces of $\boldsymbol{R}^\infty \oplus \boldsymbol{R}^\infty$ with

involution S induced by s. Then (BO_n, S) is a classifying space for n-plane bundles with involution over reasonably decent spaces with involution. The inclusion $\boldsymbol{R}^\infty \oplus \boldsymbol{R}^\infty \to \boldsymbol{R} \oplus \boldsymbol{R}^\infty \oplus \boldsymbol{R}^\infty : (x, y) \to (0, x, y)$ and identification of $\boldsymbol{R} \oplus \boldsymbol{R}^\infty$ with $\boldsymbol{R}^\infty$ (with trivial involutions) gives an equivariant inclusion $i: BO_n \to BO_{n+1}$, so that if γ_n denotes the universal n-plane bundle (with involution) $i^*(\gamma_{n+1}) \cong \gamma_n \oplus 1_+$, with 1_+ being the trivial bundle with trivial involution. Let (BO, S) be the limit of the sequence of BO_n's with these maps.

If (M, Q) is a compact n-manifold with boundary, the tangent bundle of M, as bundle with involution, is classified by a map $\tau: (M, Q) \to (BO_n, S)$. If (M, Q) is a regularly imbedded invariant submanifold of ∂V with (V, Q') an involution, then the tangent bundle of V restricts to $\tau_M \oplus 1_+$ on M, which is classified by $i \circ \tau$: $(M, Q) \to (BO_{n+1}, S)$. Thus one has a well-defined homotopy class of maps $\tau_M: (M, Q) \to (BO, S)$ classifying the stable tangent bundle of M, and if (M, Q) is regularly imbedded in the boundary of (V, Q') by $f: M \to \partial V$, then $\tau_V \circ f = \tau_M$.

Being given a pair (X, A, T) with involution, one then defines a natural transformation

$$\tau_*: \mathfrak{N}_*^{Z_2}(X, A, T) \to \mathfrak{N}_*^{Z_2}(X \times BO, A \times BO, T \times S)$$

by sending the class of $f: (M, \partial M, Q) \to (X, A, T)$ to the class of $f \times \tau_M$: $(M, \partial M, Q) \to (X \times BO, A \times BO, T \times S)$, where τ_M classifies the stable tangent bundle of M equivariantly.

The remainder of this section will be devoted to a proof of

PROPOSITION 3.1. *The composite*

$$\mathfrak{N}_*^{Z_2}(X, A, T) \xrightarrow{\tau_*} \mathfrak{N}_*^{Z_2}(X \times BO, A \times BO, T \times S) \xrightarrow{\mu} H_*^{Z_2}(X \times BO, A \times BO, T \times S; Z_2)$$

is monic.

Proof. $\mu \circ \tau_*$ is a natural transformation of equivariant homology theories, and so one has a commutative diagram

$$\begin{array}{ccc}
0 & & \\
\downarrow & & \\
\mathfrak{N}_*^{Z_2}(F, F \cap A) & \xrightarrow[\alpha]{\mu \circ \tau_*} & H_*^{Z_2}(F \times BO, (F \cap A) \times BO) \\
\downarrow i & & \downarrow i_* \\
\mathfrak{N}_*^{Z_2}(X, A) & \xrightarrow[\beta]{\mu \circ \tau_*} & H_*^{Z_2}(X \times BO, A \times BO) \\
\downarrow j & & \downarrow j_* \\
\mathfrak{N}_*^{Z_2}(X, A \cup F) & \xrightarrow[\gamma]{\mu \circ \tau_*} & H_*^{Z_2}(X \times BO, (A \cup F) \times BO) \\
\downarrow & & \\
0 & &
\end{array}$$

with $\partial: H_*^{Z_2}(X \times BO, (A \cup F) \times BO) \to H_*^{Z_2}(F \times BO, (F \cap A) \times BO)$.

Since $F\times BO$ contains the fixed set of $X\times BO$, the pair $(X\times BO, (A\cup F)\times BO)$ is relatively free, and hence

$$H_*^{Z_2}(X\times BO, (A\cup F)\times BO)\cong H_*((X\times BO)/Z_2, ((A\cup F)\times BO)/Z_2).$$

If Z is a free Z_2 space, so is $Z\times BO$ and if $\gamma_r\to Z\times BO_r$ is the pull back of the universal bundle the quotient $E(\gamma_r)/Z_2\to Z\times BO_r/Z_2$ is an r-plane bundle over $Z\times BO_r/Z_2$ and is classified by a map $\tilde{\gamma}_r\colon Z\times BO_r/Z_2\to BO_r$. The bundle 1_+ over $Z\times BO_r$ gives a trivial bundle over $Z\times BO_r/Z_2$ and one obtains a map $\tilde{\gamma}\colon Z\times BO/Z_2\to BO$. Letting $\tilde{\pi}\colon Z\times BO/Z_2\to Z/Z_2$ be induced by projection on Z, one has

$$\tilde{\pi}\times\tilde{\gamma}\colon Z\times BO/Z_2\to (Z/Z_2)\times BO$$

and this induces an isomorphism on Z_2 cohomology. (It is clearly an isomorphism on bordism for the free bordism of $Z\times BO$ may be identified with the bordism of $(Z/Z_2)\times BO$ by sending $f\times\xi\colon M\to Z\times BO$ to the class of $\bar{f}\times\bar{\xi}\colon M/Z_2\to Z/Z_2\times BO$ with $\bar{f}$ induced by f and $\bar{\xi}$ classifying $E(\xi)/Z_2\to M/Z_2$.) By excision, this also holds in the relatively free case, so

$$H_*^{Z_2}(X\times BO, (A\cup F)\times BO)\cong H_*((X/Z_2)\times BO, (A/Z_2\cup F)\times BO).$$

Now $\mathfrak{N}_*^{Z_2}(X, A\cup F)$ is isomorphic to the free bordism and so to $\mathfrak{N}_*(X/Z_2, A/Z_2\cup F)$, sending a free map $f\colon (M, \partial M)\to (X, A\cup F)$ to the class of $\bar{f}\colon (M/Z_2, \partial M/Z_2)\to (X/Z_2, A/Z_2\cup F)$. Applying τ_* and $\tilde{\pi}\times\tilde{\gamma}$ sends $(M, \partial M, f)$ to the image of the fundamental class of

$$\bar{f}\times\bar{\tau}\colon (M/Z_2, \partial M/Z_2)\to (X/Z_2\times BO, (A/Z_2\cup F)\times BO)$$

but $\bar{\tau}$ classifies the stable tangent bundle of M/Z_2. Thus, the diagram

$$\begin{array}{ccc}
\mathfrak{N}_*^{Z_2}(X, A\cup F) & \xrightarrow{\ \gamma\ } & H_*^{Z_2}(X\times BO, (A\cup F)\times BO) \\
\Big\downarrow\cong & & p\Big\downarrow\cong \\
\mathfrak{N}_*(X/Z_2, A/Z_2\cup F) & \xrightarrow{\ c\ } & H_*((X/Z_2)\times BO, (A/Z_2\cup F)\times BO)
\end{array}$$

commutes, with $c=\mu\circ\tau_*$ in ordinary bordism, which is monic. Thus γ is monic.

Next, consider the fixed set of (BO, S). Clearly, $FBO_n=\bigcup_{j+k=n} BO_j\times BO_k$ with $BO_j\times BO_k$ given by the planes in $\boldsymbol{R}^\infty\oplus\boldsymbol{R}^\infty$ of the form $U\oplus V$, $\dim U=j$, $\dim V=k$. Since $i\colon BO_n\to BO_{n+1}$ sends $BO_j\times BO_k$ into $BO_{j+1}\times BO_k$, one has $FBO=\bigcup_{k=0}^{\infty} BO\times BO_k$. Let $BO'\subset BO$ denote the fixed component $BO\times BO_0\cong BO$. One then has an equivariant inclusion

$$q\colon (X\times BO', A\times BO')\to (X\times BO, A\times BO)$$

with $F(X\times BO')=F\times BO'$. In Smith theory, one then has

$$\begin{array}{ccc} H_*^{Z_2}(X\times BO', A\times BO') & \xrightarrow{q_*} & H_*^{Z_2}(X\times BO, A\times BO) \\ \wr\| & & \downarrow pj_* \\ H_*\left(\frac{X}{Z_2}\times BO', \left(\frac{A}{Z_2}\cup F\right)\times BO'\right) & & H_*\left(\frac{X}{Z_2}\times BO, \left(\frac{A}{Z_z}\cup F\right)\times BO\right) \\ \oplus & & \\ H_*(F\times BO', (F\cap A)\times BO') & & \end{array}$$

and it is immediate that the composite pj_*q_* on the first summand is just the homomorphism induced by the map

$$\frac{Z}{Z_2}\times BO' \xrightarrow{\varkappa} \frac{Z\times BO}{Z_2} \xrightarrow{\tilde{\pi}\times\tilde{\gamma}} \frac{Z}{Z_2}\times BO$$

(on free spaces after excision). Now BO' is just a copy of BO and clearly $\tilde{\pi}\circ\varkappa$ is the projection on Z/Z_2 while $\tilde{\gamma}\circ\varkappa$ classifies the universal bundle over BO', so $(\tilde{\pi}\times\tilde{\gamma})\circ\varkappa$ is an equivalence. Thus pj_*q_* is epic, but p being an isomorphism, j_* is then epic.

The Smith theory sequence of the main diagram is then short exact, j_* being epic, with γ monic. By standard five lemma type arguments, β will be monic if α is monic.

In order to prove α is monic, one must consider $\mathfrak{N}_*^{Z_2}(F, F\cap A)$. One has an isomorphism

$$P: \textstyle\bigoplus_{k=0;k\neq 1}^{*} \mathfrak{N}_{*-k}(F\times BO_k, (F\cap A)\times BO_k) \to \mathfrak{N}_*^{Z_2}(F, F\cap A)$$

which assigns to $f: (M, \partial M)\to(F\times BO_k, (F\cap A)\times BO_k)$ the class obtained as follows. Let ξ be the k-plane bundle over M induced by $\pi_2\circ f$, $\overline{M}=D(\xi)/\{x\sim -x \mid x\in S(\xi)\}$ with involution Q induced by -1 in the fibers of ξ and with projection $\pi: \overline{M}\to M$ induced by projection in $D(\xi)$. Then $P(M, \partial M, f)$ is the class of $\pi_1\circ f\circ\pi: (\overline{M}, \partial\overline{M})\to(F, F\cap A)$. If $k=0$, $\overline{M}$ is just M with the trivial involution.

Then $H_*^{Z_2}(F\times BO, (F\cap A)\times BO)$ may be identified with the direct sum of $R=H_*(F\times BO/Z_2, (F\cap A)\times BO/Z_2\cup F\times FBO)$ and

$$S = H_*(F\times FBO, (F\cap A)\times FBO).$$

First, if $\alpha=(M, \partial M, f)\in\mathfrak{N}_*(F\times BO_0, (F\cap A)\times BO_0)$, $\mu\circ\tau_*\circ P(\alpha)$ is an element of S since $\overline{M}$ is pointwise fixed by Q. Now S decomposes as a direct sum corresponding to the components of FBO, with $\tau: \overline{M}\to FBO$ mapping into the fixed component $BO\times BO_0=BO'$. Clearly

$$\mu\circ\tau_*\circ P: \mathfrak{N}_*(F\times BO_0, (F\cap A)\times BO_0)\to H_*(F\times BO\times BO_0, (F\cap A)\times BO\times BO_0)$$

is just $\mu\circ\tau_*$ in ordinary bordism, hence is monic.

Next, if $\alpha=(M, \partial M, f) \in \mathfrak{N}_{*-k}(F\times BO_k, (F\cap A)\times BO_k)$, $k>1$, the fixed set of $\overline{M}$ is M (image of zero section, codimension k) and $RP(\xi)$ (image of $S(\xi)$, codimension 1) so that $\mu \circ \tau_* \circ P(\alpha)$ lies in the summand R. In order to evaluate $\mu \circ \tau_* \circ P(\alpha)$ one defines a boundary homomorphism

$$\partial\colon R \to H_*(F\times FBO\times BZ_2, (F\cap A)\times FBO\times BZ_2) = T.$$

Specifically, one considers $\mathfrak{N}_*^{Z_2}(F\times BO, (F\cap A)\times BO \cup F\times FBO)$ as the free bordism and takes the boundary to the free bordism

$$\mathfrak{N}_*^{Z_2}(\text{Free})(F\times FBO, (F\cap A)\times FBO)$$

which is isomorphic to $\mathfrak{N}_*(F\times FBO\times BZ_2, (F\cap A)\times FBO\times BZ_2)$. Equivalently $(F\times BO\times EZ_2, ((F\cap A)\times BO\cup F\times FBO)\times EZ_2)$ projects to

$$(F\times BO, (F\cap A)\times BO \cup F\times FBO)$$

inducing an isomorphism on Smith theory, and one applies the boundary to

$$\begin{aligned} H_*^{Z_2}(((F\cap A)\times BO \cup F\times FBO)\times EZ_2, (F\cap A)\times BO\times EZ_2) \\ \cong H_*^{Z_2}(F\times FBO\times EZ_2, (F\cap A)\times FBO\times EZ_2) \\ \cong H_*(F\times FBO\times BZ_2, (F\cap A)\times FBO\times BZ_2) = T. \end{aligned}$$

T is, of course, a direct sum over the components of FBO.

One now analyzes $\partial \circ \mu \circ \tau_* \circ P(\alpha)$. Specifically, $\tau\colon \overline{M} \to BO$ is equivariant sending M into the fixed component $BO\times BO_k$ to classify $\tau_M \oplus \xi$ (the eigenbundles) while $RP(\xi)$ maps into $BO\times BO_1$ classifying $\tau_{RP(\xi)} \oplus \lambda$ where λ is the canonical line bundle (the normal bundle in $\overline{M}=RP(\xi\oplus 1)$). Excising the fixed set in $\overline{M}$ gives a free bordism element $S(\xi)\times[0, 1]$ with involution -1×1 which maps into $F\times BO$ so that $S(\xi)\times 0 \to F\times FBO$ by projection on M composed with $\pi_1 f\times(\tau_M\times\xi)$, hence maps into $F\times BO\times BO_k$, and $S(\xi)\times 1\to F\times FBO$ is given by projection on $RP(\xi)$ composed with $(\pi_1(f\circ\pi)\times(\tau_{RP(\xi)}\times\lambda))$, hence maps into $F\times BO\times BO_1$.

Taking the boundary, and excising $S(\xi|\partial M)\times[0, 1]$ which maps into $(F\cap A)\times BO$, one has the remaining boundary given by $S(\xi)\times\{0, 1\}$ and maps as described. Thus $\partial\cdot\mu\cdot\tau_*\cdot P(\alpha)$ is the image of the fundamental classes of

$$(RP(\xi), \partial RP(\xi)) \to (F\times BO\times BO_k\times BZ_2, (F\cap A)\times BO\times BO_k\times BZ_2)$$

and

$$(RP(\xi), \partial RP(\xi)) \to (F\times BO\times BO_1\times BZ_2, (F\cap A)\times BO\times BO_1\times BZ_2)$$

with the maps given by $RP(\xi)\xrightarrow{\pi} M \xrightarrow{f\times\tau} F\times BO$, $RP(\xi)\xrightarrow{\pi} M\xrightarrow{\xi} BO_k$, $RP(\xi)\xrightarrow{\lambda} BO_1$ and $RP(\xi)\to BZ_2$ classifying the double cover by $S(\xi)$.

Letting

$$\pi_k\colon T\to H_*(F\times BO\times BO_k\times BZ_2, (F\cap A)\times BO\times BO_k\times BZ_2)$$

be the projection for the kth component, one has

$$\pi_k \circ \partial \circ \mu \circ \tau_* \circ P: \mathfrak{N}_{*-k}(F \times BO_k, (F \cap A) \times BO_k) \\ \rightarrow H_*(F \times BO \times BO_k \times BZ_2, (F \cap A) \times BO \times BO_k \times BZ_2)$$

sending $(M, \partial M, f)$ into the image of the fundamental class of $RP(\xi)$ given by the maps

$$RP(\xi) \xrightarrow{\pi} M \xrightarrow{\pi_1 f \times \tau \times \xi} F \times BO \times BO_k \quad \text{and} \quad RP(\xi) \xrightarrow{\sigma} BZ_2$$

classifying the double covers $S(\xi)$. To see that $\pi_k \circ \partial \circ \mu \circ \tau_* \circ P$ is monic, let $c \in H^1(BZ_2, Z_2)$ denote the generator, so that $H^*(RP(\xi), \partial RP(\xi); Z_2)$ is the free $H^*(M, \partial M; Z_2)$ module via π^* on $1, \sigma^*(c), \ldots, \sigma^*(c)^{k-1}$. Then for any $x \in H^*((F, F \cap A) \times BO \times BO_k)$,

$$\begin{aligned} \langle x \otimes c^{k-1}, \pi_k \partial \mu \tau_* P(\alpha) \rangle &= \langle \pi^*(\pi_1 f \times \tau \times \xi)^*(x) \cup \sigma^*(c)^{k-1}, [RP(\xi), \partial RP(\xi)] \rangle \\ &= \langle (\pi_1 f \times \tau \times \xi)^*(x), [M, \partial M] \rangle \end{aligned}$$

but the homomorphism

$$\mathfrak{N}_{*-k}(F \times BO_k, (F \cap A) \times BO_k) \rightarrow H_*(F \times BO \times BO_k, (F \cap A) \times BO \times BO_k)$$

sending $(M, \partial M, f)$ to $(\pi_1 f \times \tau \times \xi)_*[M, \partial M]$ is just $\mu \circ \tau_*$ in ordinary bordism, which is monic. Since homology and cohomology are dual, the numbers $\langle x \otimes c^{k-1}, \pi_k \partial \mu \tau_* P(\alpha) \rangle$ then determine α.

Thus, the direct sum of the homomorphisms $\pi_k \circ \partial \circ \mu \circ \tau_* \circ P$ is monic, so $\partial \circ \mu \circ \tau_* \circ P$ is monic to R on the $k \neq 0$ summand. Thus $\mu \circ \tau_* \circ P$ is monic to $R \oplus S$, and since P is epic, $\mu \circ \tau_* = \alpha$ is monic.

This completes the proof of the proposition.

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